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THERMAL SHOCK IN A DOMAIN WITH A CRACK*

V.A. KOZLOV V.G. MAZ'YA and V.Z. PARTON

Thermal shock, i.e., the action of stresses caused by an abrupt change in temperature, in a plane with a semi-infinite slit, is investigated. It is assumed that the plane has zero temperature at the initial instant, and a certain non-zero temperature acts instantaneously on the slit. The main result is the representation (5.8) of the tensile stress intensity factor SIF). The Laplace transform of the SIF is found by using a method proposed in /1/ for evaluating the coefficients in the asymptotic form of the solutions of elliptical boundary value problems near a boundary singularity. Such an approach enables one to find the SIF bypassing the solution of the initial dynamical thermoelasticity problem.

Asymptotic expressions for the SIF are derived from the formulas for the SIF, as well as for the time of fracture under instantaneous cooling of the crack edges. Thermal shock is also considered in a bounded plane domain with a rectilinear slit. It is shown that the principal term of the SIF asymptotic form for small values of time agrees with the SIF for the same problem in a plane with a slit. The expectation value of the heat shock can be obtained by the simultaneous solution of the heat conduction equation and the thermoelasticity system taking inertial terms into account. The first analytic solution of such a dynamic problem was found in /2/, where the thermal shock was considered in an elastic half-space for sudden heating of its boundary. References to other literature in the same area can be found in /3/.

1. Formulation of the boundary value problems. We will confine ourselves to considering plane strain; the case of the plane state of stress for zero heat transfer from the external medium is obtained by replacing the Lamé constant λ by $\lambda_* = 2\lambda\mu(\lambda + 2\mu)$, and γ by $\gamma_* = (1 - 2\nu)\gamma/(1 - \nu)$, where $\gamma = 2\mu\alpha_T(1 + \nu)/(1 - 2\nu)$, μ is the shear modulus, α_T is the coefficient of linear expansion, and ν is Poisson's ratio.

Let G be a plane with an excluded semi-axis $\Gamma = \{x = (x_1, x_2) : x_2 = 0,\ x_1 \leqslant 0\}$. We assume that at the initial instant G has zero temperature and the slit Γ then instantaneously acquires a constant temperature T_0. The temperature drop results in the generation of a state of stress in G and it is required to determine the stress intensity factors $K_{\rm I}(t)$ and $K_{\rm II}(t)$.

Mathematically the problem becomes the following. The temperature T is determined from the solution of the boundary value problem

$$\partial T/\partial t - a^2\Delta T = 0 \ \text{ on } G \times (0, \infty) \qquad (1.1)$$
$$T = T_0 \text{ on } \Gamma \times (0, \infty);\ T = 0 \text{ for } t = 0$$

The displacement vector u generated by this temperature field is found from the solution of the following boundary value problem:

$$-\rho\,\partial^2 u/\partial t^2 + \mu\Delta u + (\lambda + \mu)\,\mathrm{grad\,div}\, u = \gamma\,\mathrm{grad}\, T \ \text{ on } \qquad (1.2)$$
$$G \times (0, \infty)$$
$$\sigma_{22} = \gamma T,\ \sigma_{21} = 0 \ \text{ on } \Gamma \times (0, \infty)$$
$$u = \partial u/\partial t = 0 \ \text{ for } t = 0$$
$$(\sigma_{nm} = \lambda\,\mathrm{div}\, u\delta_{nm} + 2\mu\,(\partial u_n/\partial x_m + \partial u_m/\partial x_n))$$

To simplify the computations we temporarily assume that $a = 1$ and $T_0 = 1$.

2. Determination of the temperature. After a Laplace transformation, we obtain the following boundary value problem from (1.1):

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$$pT - \Delta T = 0 \text{ on } G; \ T = p^{-1} \text{ on } \Gamma$$

We seek T in the form

$$T(p, x_1, x_2) = \int_{-\infty}^{\infty} T^F(p, \xi, x_2) \exp(ix_1\xi)\, d\xi$$

Then

$$T^F(p, \xi, x_2) = \tau(p, \xi) \exp(-\sqrt{p + \xi^2}\, x_2)$$

By virtue of the evenness of the function T, we obtain that

$$\partial T / \partial x_2 = 0 \text{ for } x_2 = 0, \ x_1 > 0$$

Therefore $\tau(p, \xi)\sqrt{p + \xi^2}$ is an analytic function for $\operatorname{Im}\xi > 0$ while $\tau(p, \xi) - p^{-1}\delta(\xi)$ is an analytic function for $\operatorname{Im}\xi < 0$. Utilizing the formula

$$\delta(\xi) = \frac{1}{2\pi i(\xi - i0)} - \frac{1}{2\pi i(\xi + i0)} \tag{2.1}$$

we find

$$\tau(p, \xi) = -\frac{p^{-1/2}}{2\pi i(\xi + i0)}(p^{1/2} + i\xi)^{-1/2} \tag{2.2}$$

3. Asymptotic form of the stresses at the slit apex. It follows from /4/ that the same asymptotic formula is valid for the Laplace transform of the displacement $u(p, x_1, x_2)$ as $r \to 0$ as in the case of the plane static problem of elasticity theory. The following representation /5/ holds for the stress intensity factors $K_{\mathrm{I}}(p)$ and $K_{\mathrm{II}}(p)$

$$K_j(p) = \gamma \int_G \operatorname{grad} T\zeta^j\, dx - \gamma \int_\Gamma T\zeta_2^j\, dx_1, \quad j = \mathrm{I}, \mathrm{II} \tag{3.1}$$

where $\zeta^j(p, x)$ is the solution of the homogeneous problem

$$-\rho p^2 U + \mu \Delta U + (\lambda + \mu) \operatorname{grad} \operatorname{div} U = 0 \text{ on } G \tag{3.2}$$

$\sigma_{22} = \sigma_{21} = 0$ on Γ with the following asymptotic form as $x \to 0$:

$$\zeta^j(p, x) = Z^j(x) + O(1) \tag{3.3}$$

$$(Z_r^j, Z_\theta^j)(r, \theta) = [2(1 + \varkappa)(2\pi r)^{1/2}]^{-1} \psi^j(\theta)$$

$$\psi^{\mathrm{I}}(\theta) = ((2\varkappa + 1)\cos {}^3/_2\theta - 3\cos {}^1/_2\theta, \ (1 - 2\varkappa)\sin {}^3/_2\theta + 3\sin {}^1/_2\theta)$$

$$\psi^{\mathrm{II}}(\theta) = (\sin {}^1/_2\theta - (2\varkappa + 1)\sin {}^3/_2\theta, \ \cos {}^1/_2\theta + (1 - 2\varkappa)\cos {}^3/_2\theta)$$

where $\varkappa = 3 - 4\nu$ for plane strain and $\varkappa = (3 - \nu)/(1 + \nu)$ for the plane state of stress.

It can be verified that condition (3.3) is equivalent to the asymptotic form

$$\zeta^j(p, x) \to Z^j(x), \ p \to 0 \tag{3.4}$$

The function ζ^j are constructed below in explicit form.

Integrating by parts in (3.1), we find

$$K_j(p) = -\gamma \int_G T \operatorname{div} \zeta^j\, dx$$

4. Construction of the functions ζ^j. Let (u_1, u_2) be a solution of problem (3.2). We introduce the wave potentials φ_1, φ_2 /6/

$$u_1 = \frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2}, \quad u_2 = \frac{\partial\varphi_1}{\partial x_2} - \frac{\partial\varphi_2}{\partial x_1} \tag{4.1}$$

Then as is well-known

$$c_n^2 \Delta\varphi_n - p^2\varphi_n = 0 \text{ on } G, \ n = 1, 2 \tag{4.2}$$

$$(c_1 = \sqrt{(\lambda + 2\mu)/\rho}, \ c_2 = \sqrt{\mu/\rho})$$

The boundary conditions on the slit have the form

$$\sigma_{22} = \lambda\Delta\varphi_1 + 2\mu\left(\frac{\partial^2\varphi_1}{\partial x_2^2} - \frac{\partial^2\varphi_2}{\partial x_1 \partial x_2}\right) = 0 \text{ on } \Gamma \tag{4.3}$$

$$\sigma_{21}=\mu\left(2\frac{\partial^2\varphi_1}{\partial x_1\,\partial x_2}+\frac{\partial^2\varphi_2}{\partial x_2^2}-\frac{\partial^2\varphi_2}{\partial x_1^2}\right)=0 \quad \text{on } \Gamma$$

It follows from the asymptotic form (3.4) that the vector $U = \zeta^I$ should be subject to the conditions

$$u_n(p,\ x_1,\ -x_2) = (-1)^{n+1} u_n(p,\ x_1,\ x_2),\ n = 1,\ 2$$

The following equalities therefore result:

$$\sigma_{12}(p, x_1, -x_2) = -\sigma_{12}(p, x_1, x_2), \sigma_{nn}(p, x_1, -x_2) = \sigma_{nn}(p, x_1, x_2$$
$$\varphi_n(p,\ x_1,\ -x_2) = (-1)^{n+1}\varphi_n(p,\ x_1,\ x_2),\ n = 1,\ 2$$

Let us examine problem (4.2) in the half-space $x_2 > 0$ and let us append the condition

$$\sigma_{12} = 0,\quad \varphi_2 = 0 \quad \text{for } x_2 = 0,\quad x_1 > 0 \tag{4.4}$$

to (4.3).

We will seek the functions φ_n in the form

$$\varphi_n(p, x_1, x_2) = \int_{-\infty}^{\infty} \varphi_n^F(p, \xi, x_2)\exp(ix_1\xi)\,d\xi \tag{4.5}$$

We obtain from (4.2)

$$\varphi_n^F(p,\ \xi,\ x_2) = \Phi_n(p,\ \xi)\exp(-\eta_n x_2)\ (\eta_n = \sqrt{\xi^2 + c_n^{-2}p^2}) \tag{4.6}$$

Here that branch of the root has been selected for which the real part is positive for $\operatorname{Re} p > 0$.

As is usual, g_+ and g_- are henceforth functions that are analytic in the upper and lower half-planes, respectively, such that $g = g_+ - g_-$. We obtain from the second relationship in (4.4) that Φ_2 is continued analytically into the upper half-plane, i.e. $\Phi_2 = \Phi_{2+}$.

Because $\sigma_{12} = 0$, for $x_2 = 0$, $-\infty < x_1 < \infty$, we find

$$\Phi_1(p, \xi) = -\frac{\xi^2 + \eta_2^2}{2i\xi\eta_1}\Phi_2(p, \xi) \tag{4.7}$$

We obtain from the first relationship in (4.3)

$$(\xi^2 + \eta_2^2)\Phi_1(p,\ \xi) + 2i\xi\eta_2\Phi_2(p,\ \xi) = f_-(p,\ \xi)$$

where f_- is a function analytic for $\operatorname{Im}\xi < 0$.

Using (4.7), we arrive at the equality (R is the Rayleigh function)

$$\begin{gathered}-\frac{R(p, \xi)}{2i\xi\eta_1}\Phi_2(p, \xi) = f_-(p, \xi)\\ R(p,\ \xi) = (\xi^2 + \eta_2^2)^2 - 4\xi^2\eta_1\eta_2\end{gathered} \tag{4.8}$$

Let us present known information concerning the function R (see /6, 7/, say). The function $1/R$ has four branch points $\xi = \pm ic_1^{-1}p$, $\xi = \pm ic_2^{-1}p$ and two poles $\xi = \pm ic_R^{-1}p$, where c_R is the surface wave, the Rayleigh wave, velocity, i.e., c_R is the positive root of the equation

$$(2 - c_2^{-2}c^2)^2 - 4\sqrt{1 - c_1^{-2}c^2}\sqrt{1 - c_2^{-2}c^2} = 0$$

We set

$$D(p, \xi) = 2(c_2^{-2} - c_1^{-2})\frac{p^2(c_R^{-2}p^2 + \xi^2)}{R(p, \xi)} \tag{4.9}$$

The function D is represented in the form of the product

$$D(p, \xi) = D_+(p,\ \xi)\,D(p,\ \xi)$$

$$D_\pm(p, \xi) = \exp\left[\frac{1}{\pi}\int_{c_1^{-1}}^{c_2^{-1}}\frac{\varphi(\alpha)}{\alpha \pm i\xi/p}\,d\alpha\right]$$

$$\varphi(\alpha) = \operatorname{arctg}\frac{4\alpha^2\sqrt{c_2^{-2} - \alpha^2}\sqrt{\alpha^2 - c_1^{-2}}}{(2\alpha^2 - c_2^{-2})^2}$$

Factorization of the function η_1 has the form

$$\eta_1 = \sqrt{c_1^{-1}p + i\xi}\sqrt{c_1^{-1}p - i\xi} \tag{4.10}$$

Those branches of the roots are taken on the right for which the real part is positive for a positive real part of the radicand.

Utilizing (4.9) and (4.10) and the factorization of the function D, we find the solution of (4.8)

$$\Phi_2(p,\xi)=A\frac{i\xi}{p^2}\frac{\sqrt{c_1^{-1}p-i\xi}}{p-ic_R\xi}D_+(p,\xi) \tag{4.11}$$

Hence and from (4.7) the following formula results:

$$\Phi_1(p,\xi)=A\frac{(\xi^2+\eta_2^2)D_+(p,\xi)}{2p^2\sqrt{c_1^{-1}p+i\xi}\,(p-ic_R\xi)} \tag{4.12}$$

Substituting (4.5) and (4.6) into (4.1), we obtain

$$\begin{aligned}\zeta_1^{IF}(p,\xi,x_2)&=\Phi_2(p,\xi)\Big(\frac{\xi^2+\eta_2^2}{2\eta_1}\exp(-\eta_1x_2)-\eta_2\exp(-\eta_2x_2)\Big)\\ \zeta_2^{IF}(p,\xi,x_2)&=-\Phi_2(p,\xi)\Big(\frac{\xi^2+\eta_2^2}{2i\xi}\exp(-\eta_1x_2)+i\xi\exp(-\eta_2x_2)\Big)\end{aligned} \tag{4.13}$$

We find the value of the constant A for which the equality $\zeta^I(0,x_1,x_2)=Z^I(x_1,x_2)$ holds. We derive the following asymptotic forms from (4.13) and (4.11) as $p\to 0$:

$$\begin{aligned}\zeta_1^{IF}(p,\xi,x_2)&\sim\Phi_2(p,\xi)\Big(\frac{(c_2^{-2}-c_1^{-2})(\xi^2+\eta_2^2)p^2x_2}{4\eta_1|\xi|}-\frac{c_1^{-2}p^2}{2|\xi|^2}\Big)\times\\ &\exp(-|\xi|x_2)\sim\frac{A}{2c_R}\Big(c_1^{-2}\exp\Big(-i\frac{\pi}{4}\Big)(\xi-i0)^{-1/2}-\\ &(c_2^{-2}-c_1^{-2})x_2\exp\Big(-i\frac{\pi}{4}\Big)(\xi+i0)^{1/2}\Big)\exp(-|\xi|x_2)\end{aligned}$$

We hence obtain in $r,\ \theta$ coordinates

$$\begin{aligned}\zeta_1^I(0,\ x_1,\ x_2)&=Ac_R^{-1}\pi^{1/2}r^{-1/2}\times\\ &[{}^1/_4(5c_1^{-2}-c_2^{-2})\cos{}^1/_2\theta+{}^1/_4(c_2^{-2}-c_1^{-2})\cos{}^5/_2\theta]\end{aligned} \tag{4.14}$$

Similarly

$$\begin{aligned}\zeta_2^I(0,x_1,x_2)&=Ac_R^{-1}\pi^{1/2}r^{-1/2}/4\times\\ &[(c_1^{-2}-5c_2^{-2})\sin{}^1/_2\theta+(c_2^{-2}-c_1^{-2})\sin{}^5/_2\theta]\end{aligned} \tag{4.15}$$

Comparing (4.14) and (4.15) with the asymptotic form (3.3), we find

$$A=\frac{c_1^2c_R}{\sqrt{2\pi}}\frac{\varkappa-1}{1+\varkappa} \tag{4.16}$$

The function ζ^{II} is constructed analogously. We present the final formulas

$$\begin{aligned}\zeta_1^{IIF}(p,\xi,x_2)&=-\Psi(p,\xi)\Big(i\xi\exp(-\eta_1x_2)+\frac{\xi^2+\eta_2^2}{2i\xi}\exp(-\eta_2x_2)\Big)\\ \zeta_2^{IIF}(p,\xi,x_2)&=\Psi(p,\xi)\Big(\eta_1\exp(-\eta_1x_2)-\frac{\xi^2+\eta_2^2}{2\eta_2}\exp(-\eta_2x_2)\Big)\\ \Psi(p,\xi)&=A\frac{i\xi}{p^2}\frac{\sqrt{c_2^{-1}p-i\xi}}{p-ic_R\xi}D_+(p,\xi)\end{aligned}$$

5. An expression for the tensile stress intensity factor. By virtue of the evenness of the function T the stress intensity factor K_{II} equals zero. Applying the Parseval formula to (3.4), we find

$$\begin{aligned}K_I(p)&=-4\pi\gamma c_1^{-2}p^2\int_0^\infty\int_{-\infty}^\infty T^F(p,\xi,x_2)\varphi_1^F(p,-\xi,x_2)\,d\xi\,dx_2=\\ &-4\pi\gamma c_1^{-2}p^2\int_{-\infty}^\infty\tau(p,\xi)(\sqrt{p+\xi^2}+\eta_1)^{-1}\Phi_1(p,-\xi)\,d\xi\end{aligned} \tag{5.1}$$

Using (2.2), (4.12) and (2.1), we obtain

$$\begin{aligned}K_I(p)&=-\frac{2\pi A\gamma c_1^{-3/2}c_2^{-2}D_+(1,0)}{p^{1/2}(p^{1/2}+c_1^{-1}p)}+A\gamma c_1^{-2}p^{-3/4}\times\\ &\int_{-\infty}^\infty\frac{(p^{1/2}+i\xi)^{-1/2}}{i(\xi-i0)}(\sqrt{p+\xi^2}+\eta_1)^{-1}\frac{(\xi^2+\eta_2^2)D_+(p,-\xi)}{\sqrt{c_1^{-1}p-i\xi}\,(p+ic_R\xi)}\,d\xi\end{aligned} \tag{5.2}$$

Integration along the axis $\mathrm{Im}\,\xi = 0$ can be replaced by integration counter-clockwise around the closed contour in the half-plane $\mathrm{Im}\,\xi < 0$ and enclosing the points $-ic_1^{-1}p$ and $-ip^{1/2}$, or after replacement by the variable $i\xi = s$, by integration over the segment $(c_1^{-1}p, p^{1/2}$ from the jump of the integrand thereon. Therefore

$$K_{\mathrm{I}}(p) = -\frac{2\pi A\gamma c_1^{-3/2}c_2^{-2}D_+(1,0)}{p^{1/2}(p^{1/2}+c_1^{-1}p)} + J(p) \tag{5.3}$$

where

$$J(p) = -2\frac{A\gamma p^{-3/4}c_1^{-2}}{c_1^{-2}p^2 - p}\int_{c_1^{-1}p}^{p^{1/2}} \frac{D_+(p, is)}{s}\frac{\sqrt{p^{1/2}-s}}{\sqrt{s-c_1^{-1}p}}\frac{c_2^{-2}p^2 - 2s^2}{p - c_R s}\,ds \tag{5.4}$$

Reverting to the originals in (5.3), we conclude that

$$K_{\mathrm{I}}(t) = 2\pi A\gamma c_1^{-3/2}c_2^{-2}D_+(1,\ 0) \times \tag{5.5}$$
$$(\exp(c_1^2 t)\,\mathrm{erfc}\,(c_1 t^{1/2}) - 1) + j(t)$$

$$j(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \exp(tp)\,J(p)\,dp, \quad \alpha > 0$$

Since the function $J(p)$ is analytic in the plane with the slit along the negative axis, we convert the last integral to the form

$$j(t) = \frac{c_1^2}{2\pi i}\int_S (\exp(c_1^2 tq) - 1)\,J(c_1^2 q)\,dq$$

where the path of integration S passes over the lower edge of the negative part of the real axis and then over the upper edge. As a result of the change of variable $s = [(q^{1/2} - q)\,x + q]\,c_1$ in the integral (5.4) we obtain

$$J(c_1^2 q) = -\frac{2A\gamma c_1^{-11/2}q^{-3/4}(q^{1/2} - q)}{q^2 - q}H(-q^{-1/2})$$

Here

$$H(s) = \int_0^1 \sqrt{\frac{1-x}{x}}\,\frac{(\beta_2^2 - 2(1-(1+s)x)^2)\,Q(1-(1+s)x)}{(1+\beta_R^{-1}(1-(1+s)x))(1-(1+s)x)}\,dx \tag{5.6}$$

$$Q(\xi) = \exp\left[\frac{1}{\pi}\int_1^{\beta_2}\frac{\psi(\alpha)}{\alpha+\xi}\,d\alpha\right] \tag{5.7}$$

$$\psi(\alpha) = \mathrm{arctg}\,\frac{4\alpha^2\sqrt{\beta_2^2-\alpha^2}\sqrt{\alpha^2-1}}{(2\alpha^2-\beta_2^2)^2}$$

$$(\beta_2 = c_1c_2^{-1},\ \beta_R = c_1c_R^{-1})$$

Using the relationship $H(-s) = \overline{H(s)}$, we transform the expression for the function j to the form

$$j(t) = \frac{2\gamma}{\pi^2\beta_R c_1^{1/2}}\,\frac{\varkappa - 1}{1+\varkappa}\int_0^\infty (1 - \exp(-c_1^2 tz^2))\,z^{-3/2} \times$$
$$\mathrm{Im}\left(\frac{1+i}{z-i}H\left(\frac{i}{z}\right)\right)dz$$

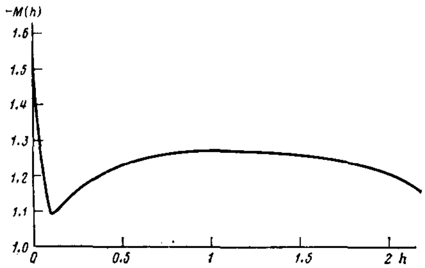
Substituting the expression obtained for j into (5.5) and removing the assumptions $a = 1$ and $T_0 = 1$, we arrive at a representation for the tensile stress intensity factor

$$K_{\mathrm{I}}(t) = T_0\gamma\,\frac{\varkappa - 1}{1+\varkappa}\,a^{1/2}t^{1/4}M(ac_1^{-1}t^{-1/2}) \tag{5.8}$$

$$M(h) = \frac{\sqrt{2}\beta_2^2 Q(0)}{\beta_R}h^{1/2}[\exp(h^{-2})\,\mathrm{erfc}(h^{-1}) - 1] + \tag{5.9}$$
$$\frac{2h^{1/2}}{\pi^2\beta_R}\int_0^\infty\left(1 - \exp\left(-\frac{z^2}{h^2}\right)\right)z^{-3/2}\,\mathrm{Im}\left(\frac{1+i}{z-i}H\left(\frac{i}{z}\right)\right)dz$$

(the functions H and Q are introduced in (5.6) and (5.7)). Note that the form of the function M is determined completely by Poisson's ratio. A graph of the function M is presented in the figure for $\nu = 0.3$.

6. The asymptotic form of $K_{\mathrm{I}}(t)$. It follows from (5.7) that Q is a smooth function for $\operatorname{Re}\xi \geqslant 0$, where



$$Q(\xi) = 1 + \frac{1}{\pi\xi}\int_{1}^{\beta_2}\psi(\alpha)\,d\alpha + O(|\xi|^{-2}), \quad |\xi| \to \infty$$

The asymptotic forms

$$H(\tau) = -\pi\beta_R + O(|\tau|^{-1}), \quad |\tau| \to \infty, \quad \operatorname{Re}\tau \geqslant 0$$
$$H(\tau) = H(0) + O(|\tau|), \quad |\tau| \to 0$$

hence follow.

Substituting the relationships obtained into (5.9), we find

$$M(h) = -\frac{4}{\pi}\Gamma\left(\frac{3}{4}\right) + O(h^{1/2}), \quad h \to 0 \tag{6.1}$$

$$M(h) = -\frac{2\sqrt{2}\beta_2^2}{\beta_R\sqrt{\pi}} h^{-1/2} + O(h^{-1}), \quad h \to \infty \tag{6.2}$$

This enables us to write the following asymptotic formulas:

$$K_{\mathrm{I}}(t) \sim -\frac{4}{\pi}\Gamma\left(\frac{3}{4}\right) T_0\gamma\frac{\varkappa-1}{1+\varkappa} a^{1/2}t^{1/4}, \quad c_1^{-1}a \ll t^{1/2} \tag{6.3}$$

$$K_{\mathrm{I}}(t) \sim -\frac{2\sqrt{2}\beta_2^2}{\beta_R\sqrt{\pi}} T_0\gamma\frac{\varkappa-1}{1+\varkappa} Q(0)(c_1t)^{1/2}, \quad c_1^{-1}a \gg t^{1/2} \tag{6.4}$$

In both cases tensile stresses originate under sudden cooling $(T_0 < 0)$, and compressive stresses under heating $(T_0 > 0)$. The first of the asymptotic forms presented is in agreement with that obtained in /5/ for the quasistatic temperature problem (i.e., for $c_1 = 0$).

In the case when $T_0 < 0$ an asymptotic form of the time t^* of the beginning of crack propagation is derived from the asymptotic form K_{I}. Indeed, t^* is the least time satisfying the equation $K_{\mathrm{I}}(t^*) = K_{IC}$, where K_{IC} is the critical value of the tensile stress intensity factor. By virtue of (5.8), this equation is equivalent to the following

$$h^{-1/2}M(h) = \left(T_0\gamma\frac{\varkappa-1}{1+\varkappa}\right)^{-1}\frac{K_{IC}c_1^{1/2}}{a}, \quad h = \frac{c_1^{-1}a}{(t^*)^{1/2}}$$

Using the asymptotic form (6.1), we find that

$$t^* \sim \left(\frac{\pi(1+\varkappa)K_{IC}}{4\Gamma(3/4)\gamma(\varkappa-1)a^{1/2}T_0}\right)^4, \quad |T_0|\gamma a c_1^{-1/2} \ll K_{IC} \tag{6.5}$$

while the relationship

$$t^* \sim \left(\frac{\sqrt{\pi}\beta_R(1+\varkappa)K_{IC}c_1^{-1/2}}{2\sqrt{2}\beta_2^2\gamma(\varkappa-1)Q(0)T_0}\right)^4, \quad |T_0|\gamma a c_1^{-1/2} \gg K_{IC} \tag{6.6}$$

follows from (6.2).

The first of these asymptotic forms agrees with the quasistatic one /5/ while the second depends substantially on the inertial term in the dynamical equations of elasticity theory. Note that the fracture time is asymptotically independent of a for large values of the thermal diffusivity a.

7. The case of a bounded domain. Let Ω_0 be a plane domain with smooth boundary Γ_0. There is a rectilinear slit l in Ω_0 that connects the origin $O(0, 0) \in \Omega_0$ with the point $A \in \Gamma_0$. We understand Γ to be the contour Γ_0 supplemented twice by the traversed segment l while Ω is the domain bounded by Γ.

The temperature Λ is determined from the solution of the boundary value problem

$$\begin{aligned} &\partial\Lambda/\partial t - a^2\Delta\Lambda = 0 \ \text{ on } \Omega \times (0, \infty) \\ &\Lambda = T_0 \ \text{ on } \Gamma \times (0, \infty), \ \Lambda = 0 \ \text{ for } t = 0 \end{aligned} \tag{7.1}$$

The displacement vector v generated by this temperature field is found from the solution of the following boundary value problem (n, τ are the normal and tangential to Γ)

$$\begin{aligned} &-\rho\partial^2 v/\partial t^2 + \mu\Delta v + (\lambda+\mu)\operatorname{grad}\operatorname{div} v = \gamma\operatorname{grad}\Lambda \ \text{ on } \Omega \times (0, \infty) \\ &\lambda \operatorname{div} v + 2\mu\partial v_n/\partial n = \gamma\Lambda \ \text{ on } \Gamma \times (0, \infty) \\ &\mu(\partial v_n/\partial\tau + \partial v_\tau/\partial n) = 0 \ \text{ on } \Gamma \times (0, \infty) \\ &v = \partial v/\partial t = 0 \ \text{ for } t = 0 \end{aligned} \tag{7.2}$$

Let $K_j^{(l)}$ ($j = \mathrm{I}$, II) be the stress intensity factors at the apex of the crack l, keeping the notation K_I for the tensile stress intensity factor in the case of a plane with a semi-infinite slit. Let s be the distance between the apex of the crack and the boundary Γ_0. We will let s be the characteristic dimension of the domain Ω.

The main information concerning $K_\mathrm{I}^{(l)}(t)$ and $K_\mathrm{II}^{(l)}$ can be obtained by combining the formula (5.8) for $K_\mathrm{I}(t)$ with the estimates

$$|K_\mathrm{I}^{(l)}(t) - K_\mathrm{I}(t)| \leqslant L, \quad |K_\mathrm{II}^{(l)}(t)| \leqslant L \tag{7.3}$$
$$L = C_N T_0 s^{1/2} \gamma (a^2 t/s^2)^N, \quad 2t \leqslant c_1^{-1} s, \quad a t^{1/2} \leqslant 1$$

where N is any positive number, the quantity C_N depends on Poisson's ratio ν, the number N and the geometry of the boundary Γ_0, but is independent of T_0, l, a, t.

In particular it therefore follows that the asymptotic formula (6.3) holds for $K_\mathrm{I}^{(l)}(t)$ in the zone $2a^2s^{-2}t < c_1^{-2}a^2t^{-1} \ll 1$ and the asymptotic formula (6.4) in the zone $a^2s^{-2}t \ll 1 \leqslant c_1^{-2}a^2t^{-1}$. The asymptotic form (6.5) for the fracture time holds for

$$|T_0| \gamma a c_1^{-1/2} \ll K_{IC} \ll |T_0| \gamma a^{1/2} c_1^{-1/4} s^{1/4}$$

and the asymptotic form (6.6) for

$$|T_0| \gamma \min\{s a^{-1} c_1^{1/2}, \ a c_1^{-1/2}\} \gg K_{IC}$$

Let us clarify how the estimates (7.3) are obtained. From dimensional analysis and because of the linearity of the problem we can confine ourselves to the case $a = 1$, $l = 1$, $T_0 = 1$, $\gamma = 1$.

Let T, u denote the solution of problem (7.1) and (7.2) in the case when Ω is identical with the plane with a slit and let us set $\Lambda_1 = \Lambda - T$, $u^{(1)} = u - v$. The stress intensity factors generated by the displacements $u^{(1)}$ will be denoted by $Q_j(t)$ ($j = \mathrm{I}, \mathrm{II}$).

We obtain the following boundary value problem for Λ_1

$$\partial\Lambda_1/\partial t - \Delta\Lambda_1 = 0 \ \text{ on } \ \Omega \times (0, \infty)$$
$$\Lambda_1 = 0 \ \text{ on } \ l, \ \Lambda_1 = 1 - T \text{ on } \Gamma_0 \times (0, \infty)$$
$$\Lambda_1 = 0 \ \text{ for } \ t = 0$$

Let B_d be a circle with centre at the point O of radius d, $2d \leqslant s$. According to the estimate obtained earlier (/5/, Sect.3) $\|\Lambda_1(t, \cdot)\|_{L_2(B_d)} \leqslant C_N t^N$. Hence, and from the known local energy estimate it follows that $\|\operatorname{grad}\Lambda_1(t, \cdot)\|_{L_2(B_d)} \leqslant C_N t^N$. Applying these estimates to $\partial^k\Lambda_1/\partial t^k$, we find

$$\|\partial^k\Lambda_1(t, \cdot)/\partial t^k\|_{L_2(B_d)} + \|\partial^k \operatorname{grad}\Lambda_1(t, \cdot)/\partial t^k\|_{L_2(B_d)} \leqslant C_N t^N \quad (k = 1, 2, \ldots)$$

The displacement vector $u^{(1)}$ satisfies the boundary value problem

$$-\rho\partial^2 u^{(1)}/\partial t^2 + \mu\Delta u^{(1)} + (\lambda + \mu)\operatorname{grad}\operatorname{div} u^{(1)} = \operatorname{grad}\Lambda_1$$
$$\sigma_{22}(u^{(1)}) = \sigma_{21}(u^{(1)}) = 0 \ \text{ on } \ l \times (0, \infty)$$
$$\sigma_{nn}(u^{(1)} - u) = -\Lambda, \quad \sigma_{n\tau}(u^{(1)} - u) = 0 \ \text{ on } \Gamma_0 \times (0, \infty)$$
$$u^{(1)} = \partial u^{(1)}/\partial t = 0 \ \text{ for } \ t = 0$$

Using the standard energy estimate for the solution of a dynamic Lamé system, we conclude that

$$\|\partial^k u^{(1)}(t, \cdot)/\partial t^k\|_{L_2(B_{d-c_1 t})} \leqslant C_N \rho^{-1} t^N \ \text{ for } c_1 t < d$$

Therefore $\mu\Delta u^{(1)} + (\lambda + \mu)\operatorname{grad}\operatorname{div} u^{(1)} = O(t^N)$ in B_{d-c_1t}. Moreover $\sigma_{22}(u^{(1)}) = \sigma_{21}(u^{(1)}) = 0$ for $x_2 = 0$, $c_1 t - d < x_1 < 0$. Applying local estimates of the solutions of elliptical Lamé systems and representations for the stress intensity in the stationary case /1, 5/, we obtain the estimate $|Q_j(t)| \leqslant C_N t^N$.

REFERENCES

1. MAZ'YA V.G. and PLAMENEVSKII B.A., On the coefficients in the asymptotic form of the solutions of the elliptical boundary value problem in domains with conical points, Math. Nachr., 76, 1977.
2. DANILOVSKAYA V.G., Temperature stresses in an elastic half-space originating because of sudden heating of its boundary, PMM, 14, 3, 1950.
3. NOWACKI W., Elasticity Theory, Mir, Moscow, 1975.
4. KONDRAT'YEV V.A., Boundary value problems for elliptic equations in domains with conical or angular points, Trudy, Moskov. Matem. Obshch., 16, 1967.
5. KOZLOV V.A., MAZ'YA V.G. and PARTON V.Z., The asymptotic form of the stress intensity factors in quasistatic temperature problems for a domain with a slit, PMM, 43, 4, 1985.
6. PARTON V.Z. and BORISKOVSKII V.G., Dynamic Fracture Mechanics. Mashinostroyenie, Moscow, 1985.
7. SLEPYAN L.I., Mechanics of Cracks, Sudostroyenie, Leningrad, 1980.

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